# Geometry of aggregates

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CHAPTER VI

# A SURVEY OF FURTHER DEVELOPMENTS

Linear systems consisting of $\infty^3$ and $\infty^4$ two-dimensional contravariant aggregates in $S_3$ have also been investigated by means of the theory developed in Chapters I, II and III. A brief survey of these investigations follows.

## § 1. The systems of aggregates $]3|2[_0 \cap S_3$.

The series $]3|2[_0 \cap S_3$ contains $\infty^1$ simply degenerated aggregates $]2[$. Then the series $]2|3[_0 \cap S_3$ resting on the former series contains $\infty^1$ simply degenerated $]3[$. The vertices of these stars $]3[$ form the manifold $\overline{]2|3[}_0 \cap S_3$ which is a curve of order 6 and genus 3, ${}^3C_1^6$. The vertices of the degenerated $]2[$ of $]3|2[_0 \cap S_3$ are trisecants of this curve and generate a surface of order 8 containing ${}^3C_1^6$ counted three times.

The series $]2|3[_0 \cap S_3$ possesses $\infty^2$ subseries $]1|3[_0 \cap S_3$. A series $]1|3[_0 \cap S_3$ rests on a manifold $\overline{]3|1[}_0 \cap S_3$ which gives a tetrahedral complex of lines. Such a complex contains as singularities four sheaves of lines and four planes of lines, the carriers of which are respectively the vertices and sides of a tetrahedron, inscribed in ${}^3C_1^6$. The surface of trisecants of ${}^3C_1^6$ turns out to be the intersection of $\infty^2$ tetrahedral complexes of lines. The planes of the $\infty^1$ simply degenerated $]2[$ of $]3|2[_0 \cap S_3$ envelop a surface of class 8. The planes containing two trisecants are the double planes of this prime-surface. Every point of ${}^3C_1^6$ belongs to 18 such planes. The above-mentioned prime-surface has a pencil of class 6 in common with every $]3[$ of $]2|3[_0 \cap S_3$. It turns out that this pencil is projectively equivalent to the curve $\overline{]2|3[}_0 \cap S_3 = {}^3C_1^6$. The birational relation between the manifolds $\overline{]2|3[}_0 \cap S_3$ and $\overline{]3|3[}_0 \cap S_2$ represents ${}^3C_1^6$ on a plane curve ${}^3C_1^4$. We obtain a clear insight into this relation if we consider it as a part of the representation of the Bordiga surface $\overline{]2|3[}_0 \cap S_4 = F_2^6$ by a plane.

Intersections of $F_2^6$ by primes give a set of $\infty^4$ curves ${}^3C_1^4$ in $S_2$ all passing through 10 base points.

Then the remaining intersection of two curves ${}^3C_1^4$ consisting of 6 points represents an intersection of ${}^3C_1^6$ by a plane. The image curve ${}^3C_1^4$ of ${}^3C_1^6$ is intersected in four points by a line; these points are the images of the four vertices of the tetrahedron of singularities belonging to a subseries $]1|3[_0 \cap S_3$ of $]2|3[_0 \cap S_3$. Moreover, one side of such a tetrahedron also contains a trisecant; thus by associating this trisecant with the fourth vertex of the tetrahedron we get a one-one correspondence between the points of ${}^3C_1^6$ and the trisecants of ${}^3C_1^6$. The birational relation between $\overline{]3|2[}_0 \cap S_3$ and $\overline{]3|2[}_0 \cap S_3^*$ affords an image of $S_3$ on $S_3^*$ whereby the points of ${}^3C_1^6$ in $S_3$ are represented on the trisecants of ${}^3C_1^6$ in $S_3^*$ and vice versa. A surface of order $n$ in $S_3$ is represented by a surface of order $3n$ in $S_3^*$.

By means of a representation of $\overline{]3|2[}_0 \cap S_3$ in the matrix-space $S_{11}$ we see that ${}^3C_1^6$ in $S_3$ is the projection of a curve of order 10 in $S_{11}$.

## § 2. The system of aggregates $]4|2[_0 \cap S_3$.

Resting on the series $]4|2[_0 \cap S_3$ we have a series $]2|4[_0 \cap S_3$. All aggregates of the latter are degenerated and 10 aggregates are two-ply degenerated; the vertices form the manifold $\overline{]2|4[}_0 \cap S_3$ consisting of 10 points $K_1, \ldots, K_{10}$.

Thus $]4|2[_0 \cap S_3$ also contains $\infty^2$ simply degenerated $]2[$, the vertices forming a congruence of lines $|d|$ of order 3 and class 6. The 10 points $K_\nu$ are the vertices of cones of order 3 of the congruence $|d|$. Paired with every two-ply degenerated $]4[$ is a regulus of lines which are the vertices of degenerated $]2[$. Every one of these reguli defines a quadratic surface containing 9 of the 10 points $K_\nu$.

We can immediately determine the mutual positions of manifolds which are connected with subseries of $]4|2[_0 \cap S_3$. The system $]2|4[_0 \cap S_3$ contains $\infty^2$ subseries $]1|4[_0 \cap S_3$. The manifold $\overline{]4|1[}_0 \cap S_3$ connected with a similar subseries consists of the entire set of lines in $S_3$ as well as a pencil of planes of class 3, the planes representing the degenerated $]1[$ of the series $]4|1[_0 \cap S_3$ which rests on $]1|4[_0 \cap S_3$. If we take a fixed $]4[$ of $]2|4[_0 \cap S_3$, then it belongs to $\infty^1$ subseries $]1|4[_0 \cap S_3$. The $\infty^1$ pencils of planes of class 3 connected with these subseries envelop a surface of class 5.

The birational relation between the manifolds $\overline{]4|2[_0} \cap S_3$ and $\overline{]3|2[_0} \cap S_4$ induces a one-one correspondence between the lines of the congruence $|d|$ and the points of the Bordiga surface $F_2^6$ in $S_4$. At the same time there is a one-one correspondence between $\overline{]2|3[_0} \cap S_4 = F_2^6$ and $\overline{]4|3[_0} \cap S_2 = S_2$ already mentioned in the preceding paragraph. Many properties of $F_2^6$ as deduced in Room (1) Chapter XIV, can now be proved by means of the above-mentioned correspondences. Conversely the Bordiga surface gives a clear representation of the congruence $|d|$ in $S_3$.

The investigation of the linear systems of two-dimensional contravariant aggregates will be complete if all systems $]n|2[_0 \cap S_3$ $(n = 1, \ldots, 10)$ have been investigated.

The theory regarding complementary systems of aggregates is of great importance in this investigation. In this case the systems are $]n|2[_0 \cap S_3$ and $]10-n|2[_0 \cap S_3$. For instance let us take $n = 8$ as an example. The system $]8|2[_0 \cap S_3$ contains a set of $\infty^2$ completely degenerated stars $]2[$. This set of planes is dual to the point-manifold $\overline{]2|2[_0} \cap S_3$. The planes envelop a surface of class 3.